$\boxed{①(a)}$ $I = (-\infty, \infty)$

$f_1(x) = x \qquad f_2(x) = 3x^2$

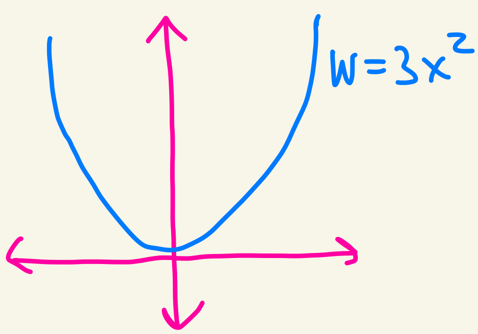
$f_1'(x) = 1 \qquad f_2'(x) = 6x$

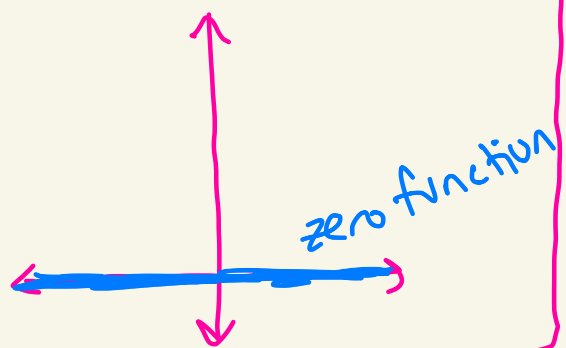
$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x & 3x^2 \\ 1 & 6x \end{vmatrix} = (x)(6x) - (3x^2)(1)$$

$$= 3x^2$$

side comments: We want to now show that the Wronskian $W(f_1, f_2) = 3x^2$ is not the zero function on $I = (-\infty, \infty)$

They differ everywhere except at $x = 0$.





There exists an x value in $I = (-\infty, \infty)$ where the Wronskian isn't equal to zero.

For example, plugging in $x = 1$ gives

$$W(f_1, f_2)(1) = 3(1)^2 = 3 \neq 0.$$

Thus, f_1 and f_2 are linearly independent on $I = (-\infty, \infty)$.

$\boxed{①(b)}$ $I = (-\infty, \infty)$

$f_1(x) = \sin(2x) \qquad f_2(x) = \sin(x)$

$f_1'(x) = 2\cos(2x) \qquad f_2'(x) = \cos(x)$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin(2x) & \sin(x) \\ 2\cos(2x) & \cos(x) \end{vmatrix}$$

$$= \sin(2x)\cos(x) - 2\sin(x)\cos(2x)$$

We want to show that this isn't equal to the zero function.

Let's find an x where $W(f_1, f_2)$ is not equal to 0.

If you try $x=0$ you will get

$$W(f_1, f_2)(0) = \underbrace{\sin(0)}_{0}\cos(0) - 2\underbrace{\sin(0)}_{0}\cos(2\cdot 0)$$

$$= 0$$

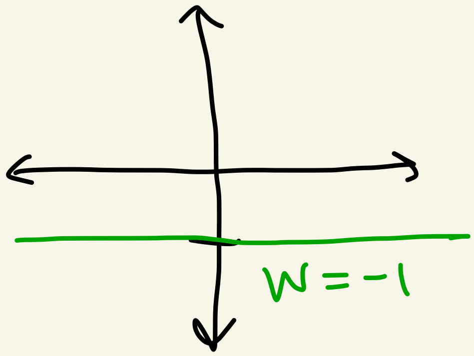
Need a different x.

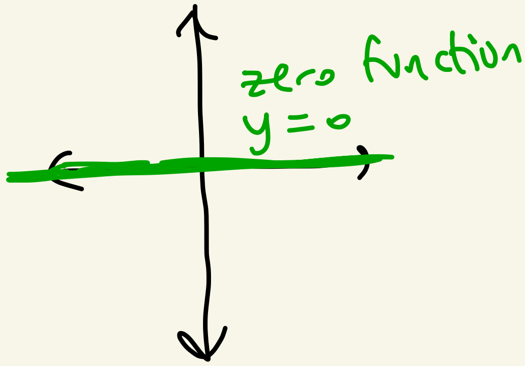
Try $x = \frac{\pi}{2}$. ← [If you try $x=\pi$ you will get 0 so it won't work]

Then

$$W(f_1, f_2)\left(\frac{\pi}{2}\right) = \underbrace{\sin\left(2\cdot\frac{\pi}{2}\right)}_{\sin(\pi)=0}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - 2\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\underbrace{\cos\left(2\cdot\frac{\pi}{2}\right)}_{\cos(\pi)=-1} = -1$$

This is definitely not the zero function.





Any x makes $W(f_1, f_2) \neq 0$.

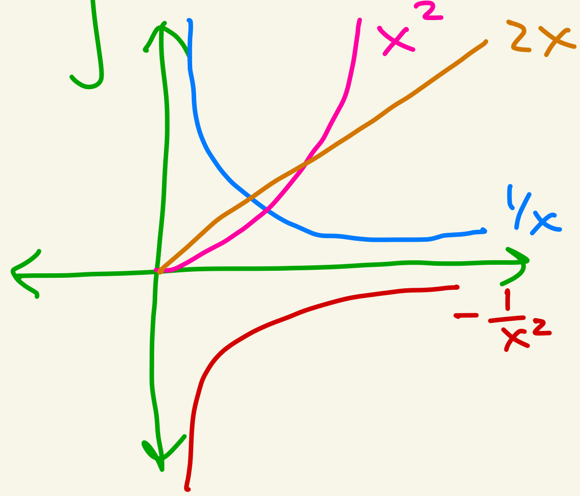
Thus, $f_1(x) = \sin(2x)$ and $f_2(x) = \sin(x)$ are linearly independent on $I = (-\infty, \infty)$.

$\boxed{①(c)}$

$f_1(x) = \frac{1}{x} = x^{-1}$ $\quad f_2(x) = x^2$

$f_1'(x) = -x^{-2}$ $\quad f_2'(x) = 2x$

} all defined on $I = (0, \infty)$



Then,

$$W(f_1, f_2) = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix}$$

$$= x^{-1} \cdot 2x - (x^2)(-x^{-2})$$

$$= 2 \cdot x^{-1+1} + x^{2-2}$$

$$= 2x^0 + x^0$$

$$= 2 + 1$$

$$= 3$$

as long as $x > 0$ then $x \neq 0$ so this makes sense

0^0 is undefined

Since the Wronskian is never zero, $f_1(x) = \frac{1}{x}$ and $f_2(x) = x^2$ are linearly independent.

$\boxed{2(a)}$ Let $y_h = c_1 x^2 + c_2 x^4$.

Let
$$f_1(x) = x^2 \quad \text{and} \quad f_2(x) = x^4.$$

Then,
$$f_1'(x) = 2x \quad \text{and} \quad f_2'(x) = 4x^3.$$
$$f_1''(x) = 2 \quad \text{and} \quad f_2''(x) = 12x^2.$$

Step 1: We must show that f_1 and f_2 are linearly independent.

We have
$$W(f_1, f_2) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = (x^2)(4x^3) - (2x)(x^4)$$
$$= 4x^{2+3} - 2x^{1+4}$$
$$= 4x^5 - 2x^5$$
$$= 2x^5$$

The Wronskian is not the zero function, for example at $x=1$, the wronskian is $W(f_1, f_2)(1) = 2(1)^5 = 2 \neq 0$.

So, f_1 and f_2 are linearly independent.

Step 2: We must show that f_1 and f_2 both solve
$$x^2 y'' - 5xy' + 8y = 0$$

This is true because plugging them into the equation gives

$$x^2 f_1'' - 5x f_1' + 8f_1 = \underbrace{x^2 \cdot (2) - 5x(2x) + 8x^2}_{0\cdot x^2} = 0$$

and

$$x^2 f_2'' - 5x f_2' + 8f_2 = \underbrace{x^2 \cdot (12x^2) - 5x(4x^3) + 8x^4}_{0\cdot x^2} = 0$$

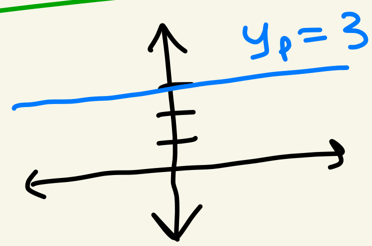
By step 1 and step 2 we know that every solution to

$$x^2 y'' - 5x y' + 8y = 0$$

is of the form

$$y_h = c_1 f_1 + c_2 f_2 = c_1 x^2 + c_2 x^4$$

2(b) Let $y_p = 3$.



Then, $y_p' = 0$, $y_p'' = 0$.

Thus, plugging y_p into the left hand side of $x^2 y'' - 5x y' + 8y = 24$ gives

$$x^2 \cdot y_p'' - 5x y_p' + 8y_p = x^2(0) - 5x(0) + 8(3) = 24.$$

So, y_p is a particular solution to $x^2 y'' - 5xy' + 8y = 24$.

2 (c) By part (a) and (b) we get that a formula for the general solution to

$$x^2y'' - 5xy' + 8y = 24$$

is

$$y = y_h + y_p = \underbrace{c_1x^2 + c_2x^4}_{y_h} + \underbrace{3}_{y_p}$$

---

2 (d) By part (c), the general solution to

$$x^2y'' - 5xy' + 8y = 24$$

is given by

$$y = c_1x^2 + c_2x^4 + 3$$

We want this solution to satisfy $y'(1) = 0$ and $y(1) = -1$

We have

$$y = c_1x^2 + c_2x^4 + 3$$
$$y' = 2c_1x + 4c_2x^3$$

So, we must solve

$$\boxed{\begin{aligned} y(1) &= -1 \\ y'(1) &= 0 \end{aligned}} \iff \boxed{\begin{aligned} c_1(1)^2 + c_2(1)^4 + 3 &= -1 \\ 2c_1(1) + 4c_2(1)^3 &= 0 \end{aligned}} \iff \boxed{\begin{aligned} c_1 + c_2 &= -4 \quad (1) \\ 2c_1 + 4c_2 &= 0 \quad (2) \end{aligned}}$$

Solve for c_1 in ① to get $c_1 = -4 - c_2$.

Plug this into ② to get $2(-4 - c_2) + 4c_2 = 0$

This gives $-8 - 2c_2 + 4c_2 = 0$.

This gives $2c_2 = 8$.

So, $\boxed{c_2 = 4}$.

Thus, $\boxed{c_1 = -4 - c_2 = -4 - 4 = 0}$.

So the solution to

$$x^2 y'' - 5xy' + 8y = 24, \quad y'(1) = 0, \quad y(1) = -1$$

is given by

$$y = \underbrace{0}_{c_1} \cdot x^2 + \underbrace{(-1)}_{c_2} \cdot x^4 + 3$$

or

$$\boxed{y = -x^4 + 3} \leftarrow \text{Answer}$$

This solution is the only solution to part (d) of this problem.

$\boxed{3(a)}$ Let $y_h = c_1 e^{2x} + c_2 x e^{2x}$.

Let $f_1(x) = e^{2x}$ and $f_2(x) = xe^{2x}$.

Then, $f_1'(x) = 2e^{2x}$ and $f_2'(x) = e^{2x} + 2xe^{2x}$

$f_1''(x) = 4e^{2x}$ and $f_2''(x) = 2e^{2x} + 2(e^{2x} + 2xe^{2x})$

$= 4e^{2x} + 4xe^{2x}$

Step 1: Show f_1 and f_2 are linearly independent

We have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x}) - (2e^{2x})(xe^{2x})$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

$e^{2x} \cdot e^{2x}$
$= e^{2x+2x}$
$= e^{4x}$

We see that the Wronskian is not the zero function. For example at $x = 0$, $W(f_1, f_2)(0) = e^{4(0)} = e^0 = 1 \neq 0$.

Thus, f_1 and f_2 are linearly independent on $I = (-\infty, \infty)$.

$\underline{\text{Step 2:}}$ Show f_1 and f_2 solve

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

This is true because

$$f_1'' - 4f_1' + 4f_1 = 4e^{2x} - 4(2e^{2x}) + 4(e^{2x}) = 0$$

and

$$f_2'' - 4f_2' + 4f_2 = 4e^{2x} + 4xe^{2x} - 4(e^{2x} + 2xe^{2x}) + 4xe^{2x}$$
$$= 4e^{2x} + 4xe^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x}$$
$$= 0$$

By step 1 and step 2 we have that the general solution to

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

is given by

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

$\boxed{3(b)}$ Let $y_p = x^2e^{2x} + x - 2$

Then, $y_p' = 2xe^{2x} + 2x^2e^{2x} + 1$

and $y_p'' = 2e^{2x} + 4xe^{2x} + 4xe^{2x} + 4x^2e^{2x}$

$$= 2e^{2x} + 8xe^{2x} + 4x^2e^{2x}$$

So, plugging y_p into the left side of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x} + 4x - 12 \quad \text{gives us}$$

$$\begin{aligned} y_p'' - 4y_p' + 4y_p &= (2e^{2x} + 8xe^{2x} + 4x^2e^{2x}) \\ &\quad - 4(2xe^{2x} + 2x^2e^{2x} + 1) \\ &\quad + 4(x^2e^{2x} + x - 2) \\ &= 2e^{2x} + x - 12 \end{aligned}$$

So, y_p solves the equation.

$\boxed{3(c)}$ By parts (a) and (b) we get that the general solution to

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x} + 4x - 12$$

is given by

$$y = y_h + y_p = \underbrace{c_1e^{2x} + c_2xe^{2x}}_{y_h} + \underbrace{x^2e^{2x} + x - 2}_{y_p}$$

$\boxed{(3)\,(d)}$ From (c) we want

$$y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x} + x - 2$$

where $y'(0) = 0$, $y(0) = 1$.

Note that

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} + 2x e^{2x} + 2x^2 e^{2x} + 1$$

We must solve

$$\boxed{\begin{aligned} y'(0) &= 0 \\ y(0) &= 1 \end{aligned}} \iff \boxed{\begin{aligned} 2c_1 e^0 + c_2 e^0 + 2c_2 \cdot 0 e^0 + 2 \cdot 0 \cdot e^0 + 2 \cdot 0^2 \cdot e^0 &= 0 \\ c_1 e^0 + c_2 \cdot 0 \cdot e^0 + 0^2 \cdot e^0 + 0 - 2 &= 1 \end{aligned}}$$

$$\iff \boxed{\begin{aligned} 2c_1 + c_2 &= 0 \\ c_1 &= 3 \end{aligned}} \iff \boxed{\begin{aligned} c_1 &= 3 \\ c_2 &= -2c_1 = -6 \end{aligned}}$$

Thus, the solution we are looking for is

$$\boxed{y = 3e^{2x} - 6xe^{2x} + x^2 e^{2x} + x - 2}$$

4 (a) Let $y_h = c_1 x^{-1/2} + c_2 x^{-1}$.

Let $f_1(x) = x^{-1/2}$ and $f_2(x) = x^{-1}$.

Then, $f_1'(x) = -\frac{1}{2}x^{-3/2}$ and $f_2'(x) = -x^{-2}$.

And, $f_1''(x) = \frac{3}{4}x^{-5/2}$ and $f_2''(x) = 2x^{-3}$.

These are all defined when $x \neq 0$ ie on $I = (0, \infty)$

<u>Step 1</u>: Show that f_1 and f_2 are linearly independent

We have that

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}$$

$$= (x^{-1/2})(-x^{-2}) - (-\tfrac{1}{2}x^{-3/2})(x^{-1})$$

$$= -x^{-1/2-2} + \tfrac{1}{2}x^{-3/2-1}$$

$$= -x^{-5/2} + \tfrac{1}{2}x^{-5/2}$$

$$= -\tfrac{1}{2}x^{-5/2}$$

This is not the zero function on I since for example at $x=1$ we get $W(f_1,f_2)(1) = -\frac{1}{2}(1)^{-5/2} = -\frac{1}{2} \neq 0$.

Thus, f_1 and f_2 are linearly independent on $I=(0,\infty)$.

$$\boxed{\begin{array}{l}\underline{\text{Step 2:}}\text{ Show that } f_1 \text{ and } f_2 \text{ solve} \\ \quad 2x^2y''+5xy'+y=0\end{array}}$$

Plugging f_1 and f_2 into the equation gives

$$\begin{aligned}2x^2f_1''+5xf_1'+f_1 &= 2x^2\left(\tfrac{3}{4}x^{-5/2}\right)+5x\left(-\tfrac{1}{2}x^{-3/2}\right)+x^{-1/2}\\ &= \tfrac{3}{2}x^{-1/2}-\tfrac{5}{2}x^{-1/2}+x^{-1/2}\\ &=0\end{aligned}$$

and

$$\begin{aligned}2x^2f_2''+5xf_2'+f_2 &= 2x^2(2x^{-3})+5x(-x^{-2})+x^{-1}\\ &= 4x^{-1}-5x^{-1}+x^{-1}\\ &=0\end{aligned}$$

So f_1 and f_2 both solve $2x^2y''+5xy'+y=0$.

By steps 1 and 2 we have that the general solution to $2x^2y''+5xy'+y=0$ is

$$y_h=c_1f_1+c_2f_2=c_1x^{-1/2}+c_2x^{-1}.$$

$\boxed{4(b)}$ Let $y_p = \frac{1}{15}x^2 - \frac{1}{6}x$.

Then, $y_p' = \frac{2}{15}x - \frac{1}{6}$

And, $y_p'' = \frac{2}{15}$.

Plugging y_p into $2x^2y'' + 5xy' + y = x^2 - x$ gives

$$
\begin{aligned}
&2x^2y_p'' + 5xy_p' + y_p \\
&= 2x^2\left(\tfrac{2}{15}\right) + 5x\left(\tfrac{2}{15}x - \tfrac{1}{6}\right) + \left(\tfrac{1}{15}x^2 - \tfrac{1}{6}x\right) \\
&= \tfrac{4}{15}x^2 + \tfrac{10}{15}x^2 - \tfrac{5}{6}x + \tfrac{1}{15}x^2 - \tfrac{1}{6}x \\
&= \tfrac{15}{15}x^2 - \tfrac{6}{6}x \\
&= x^2 - x
\end{aligned}
$$

So, y_p is a particular solution.

$\boxed{4(c)}$ The general solution to

$$2x^2y'' + 5xy' + y = x^2 - x$$

is given by

$$y = y_h + y_p = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$$

---

4 (d) We want

$$y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$$

where

$y'(1) = 0$ and $y(1) = 0$.

We have $y' = -\frac{1}{2}c_1 x^{-3/2} - c_2 x^{-2} + \frac{2}{15}x - \frac{1}{6}$

So we must solve

$$\boxed{\begin{array}{l} y'(1)=0 \\ y(1)=0 \end{array}} \iff \boxed{\begin{array}{l} -\frac{1}{2}c_1(1)^{-3/2} - c_2(1)^{-2} + \frac{2}{15}(1) - \frac{1}{6} = 0 \\ c_1(1)^{-1/2} + c_2(1)^{-1} + \frac{1}{15}(1)^2 - \frac{1}{6}(1) = 0 \end{array}}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{ll} -\frac{1}{2}c_1 - c_2 = \frac{1}{30} & (1) \\ c_1 + c_2 = \frac{1}{10} & (2) \end{array}}$$

Solving (1) for c_2 gives $c_2 = -\frac{1}{2}c_1 - \frac{1}{30}$.

Plug this into (2) gives $c_1 + \left(-\frac{1}{2}c_1 - \frac{1}{30}\right) = \frac{1}{10}$.

So, $\frac{1}{2}c_1 = \frac{2}{15}$. Thus, $\boxed{c_1 = \frac{4}{15}}$.

c_2

And, $c_2 = -\frac{1}{2}c_1 - \frac{1}{30} = -\frac{1}{2}\left(\frac{4}{15}\right) - \frac{1}{30} = \frac{-5}{30} = \boxed{-\frac{1}{6}}$

So, the solution we are looking for is

$$\boxed{y = \frac{4}{15}x^{-1/2} - \frac{1}{6}x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x}$$